# On a Series of Goldbach and Euler

**Lluís Bibiloni, Pelegrí Viader, and Jaume Paradís**

**1. INTRODUCTION.** Euler's paper *Variae observationes circa series infinitas* [**6**] ought to be considered important for several reasons. It contains the first printed version of Euler's product for the Riemann zeta-function; it definitely establishes the use of the symbol $\pi$ to denote the perimeter of the circle of diameter one; and it introduces a legion of interesting infinite products and series. The first of these is Theorem 1, which Euler says was communicated to him and proved by Goldbach in a letter (now lost):

$$\sum_{m,n\geq 2} \frac{1}{m^n - 1} = 1.$$

(One must avoid repetitions in this sum.) We refer to this result as the "Goldbach-Euler Theorem."

Goldbach and Euler's proof is a typical example of what some historians consider *a misuse of divergent series*, for it starts by assigning a "value" to the harmonic series $\sum 1/n$ and proceeds by manipulating it by substraction and replacement of other series until the desired result is reached. This unchecked use of divergent series to obtain valid results was a standard procedure in the late seventeenth and early eighteenth centuries. It has provoked quite a lot of criticism, correction, and, why not, praise of the audacity of the mathematicians of the time. They were led by Euler, the "Master of Us All," as Laplace christened him. We present the original proof of the Goldbach-Euler theorem in section 2.

Euler was obviously familiar with other instances of proofs that used divergent series. Perhaps the most well-known example was Jakob Bernoulli's proof of the divergence of the harmonic series. In this case, though, the conclusion was a contradiction instead of a positive result. As it happens, the procedure used in both proofs was the same, but it led to different answers! Despite Euler's being completely aware of this apparent paradox, he did not question Goldbach's proof of their theorem. Thus his confidence in the manipulation of the harmonic series with its infinite sum was based on something more substantial than sheer audacity.

There are quite a number of papers devoted, either directly or indirectly, to Euler's use of the infinitely large and the infinitely small. A recent one by Detlef Laugwitz is especially relevant [**12**]. Part I of this paper bears the title "The Algorithmic Thinking of Leibniz and Euler." The word "algorithmic" in the title emphasizes the point of view that Leibniz and Euler used "infinitely large" and "infinitely small" numbers like any other real number. Laugwitz argues that Euler was "more interested in algorithmic applications" than in "conceptual arguments." He was not, however, altogether careless, for in some cases it is possible to verify his results "rigorously" with the aid of the appropriate limit considerations. In Laugwitz's words [**12**, p. 450; italics added]:

> It is not difficult to verify 'rigourously' these sums of series by considering finite partial sums and passing to the limit. But for Euler *there were no limit considerations*, and what we want to know is the kind of reasoning employed by him and by his contemporaries. This being so it would be counterproductive to handle these problems using *perfected later means*.

Thus, at the base of Euler's reasoning there must have been something consistent enough to ensure that most of his results were correct. These apparently solid foundations came from a concept of number sufficiently expanded to encompass harmoniously the infinitely small and the infinitely large. Our modern model for that might be the *nonstandard analysis* introduced in Abraham Robinson's monograph [**18**]. We give a very short account of the basic ideas behind nonstandard analysis in section 3. But even these few concepts suffice to allow us to review the proof of the Goldbach-Euler theorem, in the process vindicating Goldbach and Euler's work.

In order to carry out this program, we draw upon ideas from a modern (and completely different) proof of the Goldbach-Euler theorem that appeared in this MONTHLY [**17**] as a solution to a previously proposed problem [**19**]. This proof, reproduced in section 4, is extremely short but very appealing. In the same section we examine in some detail the line of thought behind it. Not only is this an interesting exercise in its own right, but it also reveals some simple yet unexpected results to be used later in the paper. As a more recent reference we cite a different, new—though much longer—proof of the Goldbach-Euler theorem that appeared in [**1**].

We devote the rest of section 4 to the reconstruction of Goldbach and Euler's proof. We reread it both from the passage-to-the-limit point of view and from the nonstandard perspective. We show how the same arguments used by Euler, when slightly modified, become rigorous by modern standards, the main point being that almost exactly the same wording can be considered a *Weierstrassian* proof or a *nonstandard* one.

In the epilogue we show that in the same *Variae observationes* other results arrived at by means of the same techniques are not so easily amended. We also discuss how, in some instances, blunders make their way into Euler's otherwise brilliant analysis.

**2. THE GOLDBACH-EULER THEOREM.** Comment 72 from *Variae observationes circa series infinitas* [**6**] starts in a tantalizing way [**6**, p. 216; italics added] (a full translation is available at [**8**]):

> The remarks I have decided to present here generally refer to that kind of series which are *absolutely different* from the ones usually considered till now.
>
> But in the same way that to date the only series which have been considered are those whose general terms are given or, at least, the laws under which, given a few terms the rest can be found are known, I will here consider mainly those series that have neither a general term as such nor a continuation law but whose nature is determined by *other conditions*.
>
> Thus, the most astonishing feature of this kind of series would be the possibility of summing them up, as the known methods till now *require necessarily the general term* or the continuation law without which it seems obvious that we cannot find any other means of obtaining their sums.

Euler refers to a type of series whose general term is unknown and that, for this reason, had not been considered before. These series are defined by some special property of their terms and are otherwise difficult to characterize. Euler continues with his first example, the Goldbach-Euler theorem. He states [**6**, p. 216]:

> I was prompted to these remarks by a special series communicated to me by Cel. GOLDBACH whose astonishing sum, by leave of the Celebrated Master, I hereby present in the first place.

**Theorem 1.** *Consider the following series, indefinitely continued,*

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\frac{1}{31}+\frac{1}{35}+\cdots \tag{1}$$

*whose denominators, increased by one, are all the numbers which are powers of the integers, either squares or any other higher degree. Thus each term may be expressed by the formula* $\frac{1}{m^n-1}$ *where m and n are integers greater than one. The sum of this series is 1.*

The proof presented by Goldbach and Euler is the following [**6**, pp. 217–218]:

Let

$$x = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots.$$

Now, as we have

$$1 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots,$$

it will result, subtracting this series from the former

$$x-1 = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{10}+\cdots;$$

thus all powers of two, including two itself, disappear from the denominators leaving all the other numbers.

Also, if from that series above we subtract this one

$$\frac{1}{2} = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\cdots$$

there will result

$$x-1-\frac{1}{2} = 1+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{10}+\frac{1}{11}+\cdots;$$

and subtracting again

$$\frac{1}{4} = \frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots$$

it will become

$$x-1-\frac{1}{2}-\frac{1}{4} = 1+\frac{1}{6}+\frac{1}{7}+\frac{1}{10}+\cdots.$$

Proceeding similarly deleting all the terms that remain, we finally get

$$x-1-\frac{1}{2}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{9}-\cdots = 1$$

or

$$x - 1 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \cdots$$

whose denominators, increased by one, are all the numbers which are not powers. Consequently, if we subtract this series from the series we have considered at the beginning

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots ;$$

we get

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \cdots ,$$

a series whose denominators, increased by one, are all the powers of the integers and whose sum is one. ■

In the proof just offered, there are two remarkable features:

(a) the assignment of a value $x$ to $\sum 1/n$;

(b) the procedure that consists of obtaining new series (and, consequently, new results) by adding and subtracting known series and by replacing certain expressions in series with known series.

What exactly would a modern mathematician find "incorrect" or "not rigorous" in the foregoing proof?

Clearly that would be point (a), namely, the fact that Goldbach and Euler treat the harmonic series $\sum 1/n$ as if it had a real value assigned to it. If this difficulty is overlooked, point (b) can be justified by the rearrangement theorem: the adding and subtracting of series with positive terms is a common procedure when the series involved are convergent.

As we shall see, Euler was well aware of problem (a). Since the harmonic series is divergent, assigning a value to its sum is as absurd as pretending that $x := 1 + 2 + 3 + 4 + \cdots$ is a determined quantity. The real motivation for accepting the argument must have been that the procedure used led to results that were correct in the sense that they could be verified by more trustworthy finite means. A salient example is Johann Bernoulli's derivation of the sum of the telescoping series (2).

Starting with

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

and subtracting from it

$$H - 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots ,$$

Johann Bernoulli obtained

$$1 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots ,$$

which is the telescoping series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots=1. \tag{2}$$

Consequently, this procedure seems to work quite well and produce "valid" results. But there is a drawback: *in some instances, a similar manipulation leads to a contradiction*. This is the case in Jakob Bernoulli's proof of the divergence of the harmonic series. In 1689, prompted by his brother's argument, Jakob reversed it and derived the contradiction $H = H - 1$, which proved that $H$ could not be a finite quantity (see [**20**, pp. 316–323] or, with more detail, [**2**]).

We see then how, with manipulations similar to those performed in the proof of the Goldbach-Euler theorem, we arrive at a contradiction. This is caused, quite obviously, by the careless use of divergent series. As late as 1826, Abel echoed this sentiment, saying in a January 1826 letter to Holmböe [**9**, p. 16]:

> Divergent series are *in toto* an invention of the Devil, and it is a disgrace that any should venture to found on them the smallest demonstration ... if one excepts the cases of the most extreme simplicity—for example, geometric series—there is scarcely in the whole of mathematics a single infinite series of which the sum is determined in a rigorous manner.... Most of the things are exact, that is true, and it is extraordinarily surprising.

Euler knew about the divergence of the harmonic series. He acknowledged it several times in the *Variae observationes*. In 1734 he had even written a paper about it, in which, among other results, he had obtained [**5**]:

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)=\gamma, \tag{3}$$

where $\gamma$ is Euler's gamma constant (in fact, (3) can be taken as the definition of $\gamma$). Later, in his *Introductio in Analysin Infinitorum* (1748), he provided a completely different argument for the divergence of $\sum 1/n$ based on the expansion of $\log(1-x)^{-1}$ (see [**3**, pp. 29–31]).

Therefore Euler was perfectly conscious of the fragility of the method of deriving new results using divergent series. Why did he accept Goldbach's proof of the Goldbach-Euler theorem without expressing any qualms about or objection to its validity? Did he not notice the difference between divergent and convergent series, or did he simply not care?

Such is not the case. As Laugwitz points out, Euler knew the difference very well [**11**, p. 14], [**12**]. In the same paper mentioned earlier, which was dedicated to the different harmonic series (series whose general term is $c/(a+nb)$ for $n = 1, 2, \ldots$), and just before deriving (3), Euler had written [**5**, p. 88]:

> Although the terms of these series constantly decrease, if indefinitely continued the sum of the series is always infinite. In order to prove this we do not need to find a method for summing these series but the truth will easily come out from the following principle. *Series with a finite sum when indefinitely continued do not increase this sum even if continued to the double of its terms.* The quantity which is *increased behind an infinity of terms* actually remains infinitely small. If this were not the case, the sum of the series would not be determined and, con-

sequently, would not be finite. As a result of this it follows that if what remains when the terms are continued beyond the place where they begin to become infinitesimal was a finite magnitude, the sum of the series would necessarily be infinite. Therefore from this principle we may judge whether any proposed series has an infinite or a finite sum [authors' translation and italics].

These words show unmistakably that Euler gave credit to the consideration of different infinitely large and infinitely small numbers. They also hint at the fact that he could be much more explicit and accurate than he usually was in matters regarding convergence, if he were so inclined. Actually, the italicized fragment can clearly be considered as a criterion for the convergence of a series (of positive terms).

The acceptance of Goldbach's proof seems to lie in the fact that, at the time, Euler (and many of his contemporaries) actually posited a model of real numbers that included infinitely large and infinitely small numbers. Much later Bolzano [**11**, pp. 19–21] would try to put this model on a solid foundation. Today it is called the nonstandard model of the reals, a name coined by Robinson in the 1960s [**18**].

**3. A FEW STROKES OF NONSTANDARD ANALYSIS.** For a short and incisive introduction to Robinson's nonstandard analysis we recommend Lightstone's excellent paper in this MONTHLY [**13**]. Here we can outline only some of its intuitive ideas.

The fundamental notion of nonstandard analysis is the assumption of the existence of an infinitely large positive integer, call it $\Omega$, to obtain with its addition an extended number system, much in the same way as the adjoining of $i = \sqrt{-1}$ to $\mathbb{Z}$ leads to the Gaussian integers or adjoining it to the real numbers produces the complex numbers. In the case of nonstandard analysis we have to postulate a sequence of relations

$$0 < \Omega, \quad 1 < \Omega, \quad 2 < \Omega, \quad 3 < \Omega, \quad 4 < \Omega, \ldots$$

instead of the single one $i = \sqrt{-1}$, and we must make the fundamental assumption that the laws of arithmetic continue to hold.

This immediately leads to the existence of a legion of new integers such as $-\Omega$, $\Omega - 1$, $\Omega + 1$, $\Omega^2$, $\Omega^\Omega$, etc. In fact, the natural ordering must be preserved, so we get something like

$$1, 2, 3, \ldots ; \ldots , \Omega - 1, \Omega, \Omega + 1, \ldots , 2\Omega, \ldots , \Omega^2, \ldots . \tag{4}$$

Observe the use of the semicolon to separate the standard natural numbers from the nonstandard ones. If we think of $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, then we get $\mathbb{Z}(\Omega) \subset \mathbb{Q}(\Omega) \subset \mathbb{R}(\Omega)$, and we end up with the nonstandard real numbers.

In each case the guiding principle in the construction of a model of nonstandard numbers is the principle of permanence of the laws of arithmetic that goes back to Leibniz. In the jargon of nonstandard analysis it is known as *the transfer principle*, and in our naïve picture, can be stated rather vaguely as follows: *what is true for finite numbers (i.e., real numbers) is also true for infinite numbers*.

The existence of infinitely large numbers $\omega$ and our commitment to keeping the usual operations of arithmetic valid imply the existence of numbers $\epsilon = 1/\omega$ that satisfy $\epsilon < 1/n$ for all positive integers $n$. These new numbers, smaller than any positive number, are called *infinitesimals*. The existence of infinitesimals implies that each real number $a$ (we now enter the realm of real numbers) has a retinue of infinitely close nonstandard numbers, namely, $\{a \pm \epsilon : \epsilon \text{ infinitesimal}\}$.

There are a few obvious differences between standard and nonstandard numbers. For instance, between any two standard natural numbers there are only a finite number of other natural numbers, whereas between two nonstandard natural numbers there may be an infinity of other nonstandard natural numbers, for example, those between $\Omega$ and $2\Omega$ in (4).

In $\mathbb{Q}(\Omega)$ or $\mathbb{R}(\Omega)$ expressions that admit algebraic closed forms (e.g., $\sqrt{\Omega}$, $\sqrt[\Omega]{\Omega}$) are well-defined, but in dealing with infinite series we need to attach a consistent meaning to a sum that encompasses an infinity of terms. That can only be fully understood after the effective construction of a model of nonstandard real numbers that, for obvious reasons, cannot be done here. The interested reader can refer to the excellent article [**14**, pp. 57–66] for such a construction and the concept of limit in nonstandard analysis. However, the essential point is that sums with an infinity of summands can now be viewed as "ordinary" sums.

Where we used to write $\sum_{k=1}^{\infty} a_k$ to denote the limit—when it exists—of the partial sums, we now write $\sum_{k=1}^{\omega} a_k$, where $\omega$ is *any* infinitely large positive integer (the meaning of $a_k$ when $k$ is an infinite integer ought to be clear if we have a closed form for $a_k$). The catch is that now we must talk of different "sums" if different values of $\omega$ are used. If the series $\sum_k a_k$ is convergent in the standard sense, it follows from Euler's criterion (see the discussion that follows) that any two sums that extend to different infinitely large indices are not equal ($=$) but only infinitesimally close ($\approx$). For example, if $\omega_1$ and $\omega_2$ are infinite positive integers,

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\omega_1}} \approx 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\omega_2}} \approx 2.$$

If the series is divergent then this difference may be finite or even infinite, depending on the indices considered. For example, for $\omega$ any infinite positive integer,

$$1+2+3+4+\cdots+\omega = \frac{\omega(\omega+1)}{2}$$

is a true identity, though of course each side is an infinitely large number.

With these ideas in mind, Euler's criterion for convergence highlighted in the quotation of section 2 can be reworded as follows:

**Euler's Convergence Criterion.** *The series with general term $a_k$, where $a_k \geq 0$, is convergent (has a finite sum) if and only if $\sum_{k=\omega}^{2\omega} a_k$ is an infinitesimal for any infinitely large $\omega$.*

Laugwitz rephrases Euler's criterion as [**11**, p. 14]:

> A series (of real numbers) has a finite (i.e., real) sum iff the values of the sum between infinitely large numbers are infinitesimal.

He goes as far as to interpret Euler's words as the equivalent to Cauchy's 1821 convergence criterion. After an explicit construction of a nonstandard model for the real numbers, these criteria are provable propositions.

**Remark.** With Laugwitz's more general formulation, if we consider the sum $\sum_{k=\omega_1}^{\omega_2} a_k$ for any two different infinitely large numbers $\omega_1$ and $\omega_2$ instead of $\sum_{k=\omega}^{2\omega} a_k$, Euler's criterion remains valid for series of arbitrary terms.

The application of Euler's criterion to the harmonic series,

$$\sum_{k=\Omega+1}^{2\Omega} \frac{1}{k} = \frac{1}{\Omega+1}+\frac{1}{\Omega+2}+\cdots+\frac{1}{2\Omega} > \frac{\Omega}{2\Omega} = \frac{1}{2},$$

immediately proves its divergence. In fact, we can easily go a little further and see that

$$\sum_{k=\Omega+1}^{2\Omega} \frac{1}{k} \approx \log 2.$$

Indeed, if we define the $n$th harmonic number as

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \tag{5}$$

the nonstandard version of (3) is

$$H_\Omega \approx \log \Omega + \gamma \tag{6}$$

(Euler writes $=$), and it leads to

$$\sum_{k=\Omega+1}^{2\Omega} \frac{1}{k} \approx \sum_{k=1}^{2\Omega} \frac{1}{k} - \sum_{k=1}^{\Omega} \frac{1}{k} \approx \log 2\Omega - \log \Omega \approx \log 2.$$

All these considerations are the result of the modern and rigorous point of view offered by nonstandard analysis, but we must concede that they resonate well with Euler's views!

Unfortunately, we cannot deal more fully with nonstandard analysis in this article. Still, we hope that the few glimpses of the subject offered here will arouse the reader's interest in getting better acquainted with its methods, in the spirit of André Weil's statement that [**7**, p. xii]:

> [O]ur students of mathematics would profit much more from the study of Euler's *Introductio in Analysin Infinitorum*, rather than of the available modern text-books.

**4. GOLDBACH AND EULER'S PROOF REVISITED.** The Goldbach-Euler theorem appears as problem 132 in the excellent book by Konrad Knopp [**10**, p. 273]. It is there that the authors first came across it and then proposed it at a problem-solving seminar at the Catalan Mathematical Society. As we later discovered, the simple and elegant proof we had found at the seminar already existed (like almost everything else in mathematics!). It had appeared in this MONTHLY [**17**, pp. 402–403]. We reproduce the proof here for the sake of completeness.

**Solution of the Goldbach-Euler theorem by University of South Alabama Problem Group.** Let $S$ be the set of positive integers that are powers, and let $T$ be its complement, that is, the nonpowers. Then

$$\sum_{s\in S}(s-1)^{-1} = \sum_{k\geq 2}\sum_{a\in T}(a^k-1)^{-1} = \sum_{k\geq 2}\sum_{a\in T}\sum_{i\geq 1} a^{-ik}$$
$$= \sum_{n\geq 2}\sum_{k\geq 2} n^{-k} = \sum_{n\geq 2}(n(n-1))^{-1} = 1. \qquad \blacksquare$$

This short proof is based on the well-known elementary identity

$$\frac{1}{n-1} = \frac{1}{n} + \frac{1}{n(n-1)}, \tag{7}$$

which we refer to as the *telescoping identity*.

The terms in the Goldbach-Euler series (1) are of the form

$$\frac{1}{a^m - 1}. \tag{8}$$

The difficulty in obtaining the sum of the series arises from the fact that, as Euler pointed out, we cannot use (8) as its general term. But if we apply the telescoping identity (7) to (8) with $n = a^m$ we have

$$\frac{1}{a^m - 1} = \frac{1}{a^m} + \frac{1}{a^m(a^m - 1)}. \tag{9}$$

In this way (8) splits into two summands of different “shapes”: $1/n$ and $1/[n(n-1)]$.

To obtain our sum, we let $a$ run over the integers that are nonpowers, and we let $m \geq 2$:

$$\sum_a \sum_{m\geq 2} \frac{1}{a^m - 1} = \sum_a \sum_{m\geq 2} \frac{1}{a^m} + \sum_a \sum_{m\geq 2} \frac{1}{a^m(a^m - 1)}. \tag{10}$$

For a fixed $a$, the inner sum in the first summand in (9) gives rise to a geometric series:

$$\sum_{m\geq 2} \frac{1}{a^m} = \frac{1}{a(a-1)}. \tag{11}$$

After this, quite unexpectedly, both summands in (10) assume the same shape. One is

$$\sum_a \frac{1}{a(a-1)},$$

where $a$ runs over the integers that are nonpowers, and the other is

$$\sum_a \sum_{m\geq 2} \frac{1}{a^m(a^m - 1)} =: \sum_b \frac{1}{b(b-1)},$$

where $b$ runs over the integers that are powers. After these elementary transformations, the series in Goldbach-Euler theorem can be expressed in the form

$$\sum_a \frac{1}{a(a-1)} + \sum_b \frac{1}{b(b-1)}. \tag{12}$$

Now, as the $a$s and the $b$s taken together comprise all integers greater than 1 without repetitions, (12) reduces to Bernoulli’s telescoping series (2), whose sum we know to be 1.

To complete a rigorous proof, the reader need only use the rearrangement theorem for convergent series of positive terms and two well-known elementary facts:

(a) the sum of the geometric series (11), which Goldbach and Euler also use in their proof,

and

(b) the sum of the telescoping series (2) in the last step.

It is interesting to observe that these two results can be derived by iteration of the telescoping identity (7) in two different ways that are totally within Euler's spirit of formal manipulation. Besides having some classroom value, this point of view will be useful in what follows.

First, upon substitution of $n-1$ for $n$ in the telescoping identity we get

$$\frac{1}{n-1} = \frac{1}{(n-1)n} + \frac{1}{n} = \frac{1}{(n-1)n} + \frac{1}{n(n+1)} + \frac{1}{n+1}$$

and iteration leads to

$$\frac{1}{n-1} = \frac{1}{(n-1)n} + \frac{1}{n(n+1)} + \cdots + \frac{1}{(n+m-1)(n+m)} + \frac{1}{n+m}$$

for any positive integer $m$. We can then state:

**Lemma 1.** *For any positive integers $n$ and $k$ with $2 \le n < k$*

$$\frac{1}{n-1} = \frac{1}{(n-1)n} + \frac{1}{n(n+1)} + \cdots + \frac{1}{(k-1)\,k} + \frac{1}{k}.$$

Second, the same procedure iterating on $1/(n-1)$ leads to

$$\frac{1}{n-1} = \frac{1}{n} + \frac{1}{n}\frac{1}{(n-1)} = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^2(n-1)},$$

which gives rise to a second item of note:

**Lemma 2.** *For any positive integers $n$ and $k$ with $n \ge 2$*

$$\frac{1}{n-1} = \frac{1}{n} + \frac{1}{n^2} + \cdots + \frac{1}{n^k} + \frac{1}{n^k(n-1)}.$$

We are finally prepared to reconsider Goldbach and Euler's proof of the Goldbach-Euler theorem. We will see how, with almost the same phrasing, the proof can be made rigorous from both the standard and the nonstandard vantage points.

As in (5), we let $H_{\boldsymbol{n}}$ denote the $\boldsymbol{n}$th harmonic number, but we now think of $\boldsymbol{n}$ as either a finite natural number or an infinite nonstandard natural number. Let $k_2$ be defined by $2^{k_2} \le \boldsymbol{n} < 2^{k_2+1}$. The existence and uniqueness of $k_2$ is clear either if we think of $\boldsymbol{n}$ as a finite natural number or as a nonstandard natural number: remember the transfer principle! Using Lemma 2, we can write

$$1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{k_2}} + \frac{1}{2^{k_2}\cdot 1},$$

and subtracting this series from (5), we obtain

$$H_{\boldsymbol{n}} - 1 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \cdots + \frac{1}{\boldsymbol{n}} - \left(\frac{1}{2^{k_2}\cdot 1}\right). \tag{13}$$

Hence, all powers of two, including two itself, disappear from the denominators, leaving the rest of integers up to $\boldsymbol{n}$.

If from (13) we subtract

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{k_3}} + \frac{1}{3^{k_3} \cdot 2},$$

again obtained from Lemma 2 with $k_3$ defined by $3^{k_3} \le \boldsymbol{n} < 3^{k_3+1}$, the result will be

$$H_{\boldsymbol{n}} - 1 - \frac{1}{2} = 1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \cdots + \frac{1}{\boldsymbol{n}} - \left( \frac{1}{2^{k_2} \cdot 1} + \frac{1}{3^{k_3} \cdot 2} \right).$$

Proceeding similarly we end up by deleting all the terms that remain, arriving finally at

$$H_{\boldsymbol{n}} - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{10} - \cdots - \frac{1}{\boldsymbol{n}} = 1 - \left( \frac{1}{2^{k_2} \cdot 1} + \frac{1}{3^{k_3} \cdot 2} + \cdots + \frac{1}{\boldsymbol{n} \cdot (\boldsymbol{n} - 1)} \right). \tag{14}$$

(Notice that $k_2 \ge k_3 \ge \cdots$. In fact, when $m > \sqrt{\boldsymbol{n}}$ we get $k_m = 1$.) This last expression has been obtained assuming that $\boldsymbol{n}$ is a nonpower. If $\boldsymbol{n}$ is a power, then $1/\boldsymbol{n}$ will have disappeared at some stage of this process, and the last fraction to be removed from (13) will be $1/(\boldsymbol{n} - 1)$, whose denominator is a nonpower unless $\boldsymbol{n} = 9$. (This is Catalan's conjecture that 8 and 9 are the only consecutive powers that exist. The conjecture was recently proved by Mihăilescu [**15**], [**16**]. In fact, it does not matter here whether there are more consecutive powers or not.) The corresponding expression will thus be

$$H_{\boldsymbol{n}} - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{10} - \cdots - \frac{1}{\boldsymbol{n} - 1} = 1 - \left( \frac{1}{2^{k_2} \cdot 1} + \frac{1}{3^{k_3} \cdot 2} + \cdots + \frac{1}{(\boldsymbol{n} - 1) \cdot (\boldsymbol{n} - 2)} \right). \tag{15}$$

Consequently, if we subtract (14) from (5) we obtain

$$1 - \left( \frac{1}{2^{k_2} \cdot 1} + \frac{1}{3^{k_3} \cdot 2} + \cdots + \frac{1}{\boldsymbol{n} \cdot (\boldsymbol{n} - 1)} \right) = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \cdots + \frac{1}{\boldsymbol{n} - 1}$$

or, correspondingly subtracting (15) from (5),

$$1 - \left( \frac{1}{2^{k_2} \cdot 1} + \frac{1}{3^{k_3} \cdot 2} + \cdots + \frac{1}{(\boldsymbol{n} - 1)(\boldsymbol{n} - 2)} \right) = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \cdots + \frac{1}{\boldsymbol{n}},$$

sums that contain in their denominators, increased by one, all the powers of the integers up to $\boldsymbol{n}$. We must now take care of the "remainder," that is, the expression between parentheses above or on the right-hand side of (14) (respectively, (15)).

Since for each $m(\geq 2)$ we know by the definition of $k_m$ that $\boldsymbol{n} < m^{k_m+1} \leq m^{2k_m}$, it follows that $\sqrt{\boldsymbol{n}} < m^{k_m}$ and

$$\frac{1}{m^{k_m} \cdot (m-1)} \leq \frac{1}{\sqrt{\boldsymbol{n}}} \cdot \frac{1}{m-1}.$$

This implies that

$$\frac{1}{2^{k_2} \cdot 1} + \frac{1}{3^{k_3} \cdot 2} + \cdots + \frac{1}{\boldsymbol{n} \cdot (\boldsymbol{n}-1)} \leq \frac{H_{\boldsymbol{n}-1}}{\sqrt{\boldsymbol{n}}}$$

or, if $\boldsymbol{n}$ is a power,

$$\frac{1}{2^{k_2} \cdot 1} + \frac{1}{3^{k_3} \cdot 2} + \cdots + \frac{1}{(\boldsymbol{n}-1) \cdot (\boldsymbol{n}-2)} \leq \frac{H_{\boldsymbol{n}-2}}{\sqrt{\boldsymbol{n}-1}}.$$

If we have chosen to regard $\boldsymbol{n}$ as a finite integer, say $\boldsymbol{n} = n$, then we can pass to the limit and use Euler's asymptotic value (3) for $H_n$

$$\lim_{n \to \infty} \frac{H_{n-1}}{\sqrt{n}} = \lim_{n \to \infty} \frac{\log(n-1) + \gamma}{\sqrt{n}} = 0.$$

The "standard" proof is now complete.

But if we are willing to believe in infinite integers and infinitesimals we do not need to pass to the limit. We again use (3) but now as a nonstandard equality:

$$\frac{H_{\boldsymbol{n}-1}}{\sqrt{\boldsymbol{n}}} \approx \frac{\log(\boldsymbol{n}-1) + \gamma}{\sqrt{\boldsymbol{n}}} \approx 0,$$

which yields

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \cdots + \frac{1}{\boldsymbol{n}-1} \approx 1.$$

In this case, what we actually obtain is a different nonstandard number infinitesimally close to 1 for each infinite $\boldsymbol{n}$ considered.

**5. EPILOGUE.** Unfortunately, not all of Euler's maneuvers with infinite numbers and infinitesimals are so easily amended. This is the case for many of the results in the second part of the *Variae observationes* (for all the references to the *Variae*, see [**6**]). This part deals mainly with Euler's product formula for the Riemann zeta-function, which appears in Theorem 8 [**6**, p. 230]:

$$\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots = \zeta(s) \tag{16}$$

if $s > 1$. Euler treats only the case where $s$ is a positive integer.

The proof of (16) follows the same lines as the proof for the special case $s = 1$—this is where the real trouble lies—and constitutes Theorem 7 [**6**, p. 227]:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots = \prod_{p \text{ prime}} \frac{1}{1-p^{-1}}. \tag{17}$$

Euler clarifies the meaning of (17) by pointing out that both sides have value infinity. In this particular case keeping track of the remainders is much more difficult (if at all feasible). That may be the reason why Euler is so ambiguous in the first part of the paper when he could have been much more rigorous. He proceeds as follows. From

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Euler obtains

$$\frac{1}{2}x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots ,$$

which upon subtraction gives

$$\left(1 - \frac{1}{2}\right)x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots ,$$

a series with no even denominators. Multiplying this series by $1/3$ and subtracting the previous result from it, he gets

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdot x = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots ,$$

and so on until all proper fractions on the right-hand side are deleted. Euler concludes that

$$x \cdot \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right) = 1,$$

which he equates with (17).

If now we consider (6), Corollary 1 to Theorem 7 establishes the "degree" of infinitude of the harmonic series [**6**, p. 229]:

> [I]f we denote absolute infinity as $\infty$, then the value of this expression [ $\prod_p \frac{1}{1-p^{-1}}$] is $\log \infty$, which is the minimum among all powers of infinity.

Lastly, by taking logarithms in (17) Euler derives Theorem 19, the closing theorem of the *Variae*, the divergence of the series whose terms are the reciprocals of the primes [**6**, p. 242]:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \log\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots\right)$$
$$= \log\log\infty. \qquad (18)$$

Euler's carelessness with the different meanings he attaches to the equality sign is remarkable. As they stand, (17) and (18) are unacceptable. To render them correct it is necessary to replace the infinite series with the corresponding $n$th partial sums and each equality sign with $\sim$ with the usual meaning that the quotient of the two sides goes to 1 as $n$ goes to infinity. In fact, the strong form of both these theorems is due to Mertens in 1874 (see [**4**, p. 6, footnote]) and can be stated as follows (with $p$ prime and $\omega$ any infinitely large integer):

$$\lim_{n\to\infty} \prod_{p \le n} \left(1 - \frac{1}{p}\right) \cdot H_n = e^{-\gamma}$$

or

$$\prod_{p \le \omega} \left(1 - \frac{1}{p}\right) \cdot H_\omega \approx e^{-\gamma},$$

and

$$\lim_{n \to \infty} \left( \sum_{p \le n} \frac{1}{p} - \log H_n \right) = \gamma + B,$$

or

$$\sum_{p \le \omega} \frac{1}{p} \approx \log H_\omega + \gamma + B,$$

where $B = \sum_p \left(\log(1 - p^{-1}) + p^{-1}\right)$.

As a final comment we want to add that, on occasion, Euler's carelessness went too far. A striking example is Theorem 18 [**6**, p. 241], where he claimed to have proved that $\sum_{n=1}^{\infty} \lambda(n)/n = 0$. Here $\lambda$ is Liouville's function ($\lambda(n) = (-1)^{r(n)}$, in which $r(n)$ is the number of prime divisors of $n$ counted according to multiplicity). This result is true, but it is as deep (and as difficult to prove) as the Prime Number Theorem!

**ACKNOWLEDGMENTS.** We are very grateful to our colleague Angel Gil of Universitat Pompeu Fabra for his careful reading of our manuscript and his many valuable suggestions. We also thank Isabel Tornero for correcting the English.

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**LLUÍS BIBILONI** is an associate professor of mathematics education at the Departament de Didàctica de les Matemàtiques i de les Ciències Experimentals of the Universitat Autònoma de Barcelona. He received his Ph.D. in mathematics from the Universitat Politècnica de Catalunya. As to mathematics proper, his main research interest is the metrical theory of the continuum and related subjects. He has been deeply influenced by the work of George Pólya on problem solving and plausible reasoning, and he is pursuing Pólya's ideas at the practical teaching level as well as at the research level in mathematics education. His favorite mathematical subjects are number theory, mathematical logic, nonstandard analysis, and the philosophy and history of mathematics.
*Facultat de Ciències de l'Educació, Univ. Autònoma Barcelona, 08193 Bellaterra, Barcelona, Spain.*
*lluis.bibiloni@uab.es*

**PELEGRÍ VIADER** is an associate professor of mathematics at the Universitat Pompeu Fabra, Barcelona. He received his Ph.D. in mathematics from the Universitat Politècnica de Catalunya. His main research interests are algorithms for the representation of real numbers, singular functions, and the history of mathematics. His interests besides mathematics are history at large, tennis, and enjoying the company of family and friends over a nice *paella* on a Mediterranean beach.
*Departament d'Economia i Empresa, Universitat Pompeu Fabra, Ramon Trias Fargas 25–27, 08005 Barcelona, Spain.*
*pelegri.viader@upf.edu*

**JAUME PARADÍS** received his Ph.D. in mathematics from the Universitat Politècnica de Catalunya. He is now an associate professor at the Universitat Pompeu Fabra, Barcelona. His early research includes several books and papers on the history of algebra, mainly on Viète's and Fermat's contributions. He focuses his current research on Pierce expansions and other systems of real number representation and on singular functions defined with their help.
*Departament d'Economia i Empresa, Universitat Pompeu Fabra, Ramon Trias Fargas 25–27, 08005 Barcelona, Spain.*
*jaume.paradis@upf.edu*